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LETTER TO THE EDITOR

Some asymptotic estimates in the random parking problem

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Abstract. As shown by exact calculations in one dimension ($d=1$) and by computer experiments for $d=2$, the density of the jammed state in the random parking problem tends to its limit value as $y_d(\infty)-a_d\tau^{-1/d}$, where τ is the time and $y_d(\infty)$ the final density. The pair correlation function diverges at contact in the final state. Both properties are due to the sure filling of small holes.

The one-dimensional (1D) parking problem, as posed by Bernal, is the following (Rényi 1958): take a 'car' of length l and put it at random on a line, then put another car of length l at random in the remaining space and so on. The process terminates when no hole of length larger than l remains between two cars. The distribution which is obtained in this way is *not* the Gibbs distribution for hard rods on a line, since in this Gibbs ensemble, at any number density smaller than the close packing, the holes are Poisson distributed and can be larger than any fixed length.

The dynamics of the 1D filling process has been studied by Gonzáles *et al* (1974) and we shall briefly quote here part of their results.

The natural unit of length is the length of a car, say l, so that the unit of number density will be l^{-1}. Furthermore, the dynamics of the filling process enters as follows: one tries to insert cars at random on the line at a rate R per unit length and time. Since the attempts for inserting new cars are made at random, sometimes one finds an already occupied place and must try again. Of course checks become more and more frequent as time goes on. The unit time is $(Rl)^{-1}$. With these dimensionless units, the number density at time τ is

$$y_1(\tau)=\int_0^\tau \mathrm{d}u\,\exp\left(-2\int_0^u v^{-1}(1-\mathrm{e}^{-v})\,\mathrm{d}v\right)$$

where the subscript '1' refers to the 1D character of the problem. As τ goes to infinity one recovers the limit density found by Rényi:

$$y_1(\infty)=\int_0^\infty \mathrm{d}u\,\exp\left(-2\int_0^u v^{-1}(1-\mathrm{e}^{-v})\,\mathrm{d}v\right)\simeq 0{\cdot}7476\ldots.$$

Near $\tau=\infty$, $y_1(\tau)$ expands as

$$y_1(\tau)\underset{\tau\to\infty}{\simeq} y_c(\infty)-\frac{\mathrm{e}^{-2\gamma}}{\tau}+4\frac{\mathrm{e}^{-2\gamma}\ln\tau\, e^{-\tau}}{\tau^2}+\ldots \tag{1}$$

where γ is Euler's constant.

The same problem has been examined in two dimensions, with either oriented hard squares or hard discs (Finegold and Donnell 1979). No analytical solution seems to be known in these two cases. On numerical evidence Feder (1979) noticed that, at the end of the covering process with hard discs, the number density tends to its asymptotic value as

$$y_2(\tau) \simeq y_2(\infty) - a_2\tau^{-1/2} + \ldots \tag{2}$$

where again the subscript 2 refers to the dimensionality.

From (1) and (2), it is tempting to speculate that the asymptotic behaviour of $y_d(\tau)$ at any dimensionality d is

$$y_d(\tau) \simeq y_d(\infty) - a_d\tau^{-1/d} + \ldots \tag{3}$$

with $a_d > 0$. In the present Letter, we show that (3) is indeed true when the parked cars are spheres, as we shall assume from now on (that is, hard rods in 1D, hard discs in 2D, etc). Furthermore, we show by a slight extension of the arguments that the pair correlation function diverges logarithmically near the contact.

For this proof, we use the notion of the Voronoi–Dirichlet polyhedron, which is defined as follows. Let $\mathscr{P} = \{P_j; P_j \neq P_k \text{ unless } j = k, j \in \mathbb{N}^*\}$ be a discrete point set in $\mathbb{R}^d$. The Voronoi–Dirichlet polyhedron of $P_i (i \in \mathbb{N}^*)$ is the (convex) polyhedron made of all points in $\mathbb{R}^d$ closer to P_i than to any other $P_j \in \mathscr{P}$, $P_j \neq P_i$. Unless the configuration of points $\mathscr{P}$ is a very special one, and we shall assume that this is *not* the case in the random parking problem, no point in $\mathbb{R}^d$ is at the same distance from more than $(d+1)$ points in $\mathscr{P}$. At $d = 2$ this implies that three edges meet at each vertex of a Voronoi–Dirichlet polyhedron, this vertex being equidistant from three points in $\mathscr{P}$.

Another ingredient for explaining (3) is the recognition that the dynamics of the filling process proceeds following two steps. Firstly, during the 'configuration building' step all the large holes are filled up. This first step progresses exponentially to its end: consider holes large enough so that once a car fills one, it is still possible to put in at least one more car. Unless we have very unlikely conditions, the volume where the centre of the first car can lie is bounded from below by a quantity of order l^d, where l is the car diameter. By random trials, the number density of these large holes decays at least as $\exp[-(kRl^dt)]$, where kl^d is some lower bound for the volume of these large holes, k being some constant and R the parking rate per unit volume and time.

To obtain the algebraic time convergence to the jamming density, one assumes that this configuration building is ended and considers the filling of small holes only; these holes are such that, once filled, no other car can be put in them.

At the end of the filling process, one puts the centres of the cars in small regions around the vertices of the Voronoi–Dirichlet polyhedra (VVDP) built up from the set $\mathscr{P}$ of centres of the already parked cars. These vertices are distant little more than $2l$ from the centres of the $(d+1)$ neighbouring cars. To define more precisely the local situation around these vertices, let ρ be the distance from one of these vertices to the centre of the adjacent cars, and Θ be the set of angles (two angles in $d = 2$) defining, for instance, the mutual orientations of the edges meeting at the vertex. The number density of VVDP in $\mathrm{d}\rho\, \mathrm{d}\Theta$ is by definition $p(\rho, \theta)\, \mathrm{d}\rho\, \mathrm{d}\Theta$, the whole number density of these vertices being $\int \mathrm{d}\rho\, \mathrm{d}\Theta p(\rho, \Theta)$.

If ρ is a little larger than $2l$, say $\rho = 2l + \delta$, with $\delta \ll l$, the allowed region for inserting the centre of a new car around a VVDP is a small simplex of volume $\sigma = \delta^d f_d(\Theta)$, where $f_d(\,.\,)$ depends on angles only, although δ^d is a scale factor for size of this simplex. For instance, δ can be thought as the radius of the inscribed sphere in this simplex.

Thus $\mathrm{d}\rho \int \mathrm{d}\Theta p(\rho, \Theta)$ is the number density of holes with ρ in $[\rho, \rho+\mathrm{d}\rho]$. As $\sigma = \delta^d f_d(\Theta)$, the number density of holes with a volume in $[\sigma, \sigma+\mathrm{d}\sigma]$ is $q(\sigma)\,\mathrm{d}\sigma$, where

$$q(\sigma) = \sigma^{(1/d)-1} A_d,$$

with

$$A_d \equiv \int \frac{\mathrm{d}\Theta}{d} f^{-1/d} p(2l|_-, \Theta).$$

We have used $p(2l|_-, \Theta)$ in this formula, as we are interested in small holes only ($\delta \sim 0$). As $p(\rho, \Theta)$ is continuous at $\rho = 2l$ for any finite time, one may replace $p(2l+\delta, \Theta)$, $\delta > 0$, by its value extrapolated from $\rho \leqslant 2l$, that is by $p(2l|_-, \Theta)$, as this last quantity can be measured after the end of the filling process and is not zero, although $p(\rho, \Theta)$ is obviously zero for $\rho > 2l$, at the end of the proces. Due to the random filling of small holes, $q(\sigma)$ depends on time as $q(\sigma, t^*) \exp[-R\sigma(t-t^*)]$, where t^* is some finite time such that the configuration building is practically ended at $t > t^*$. Thus the number density of remaining holes at time t is $Q(t) = \int_0^{\sigma^*} \mathrm{d}\sigma q(\sigma, t)$, where σ^* is some upper bound for the volume of a 'small' hole. From $q(\sigma, t) \simeq \sigma^{(1/d)-1} A_d \exp[-R\sigma(t-t^*)]$:

$$Q(t) \underset{t\to\infty}{\simeq} \int_0^{\sigma^*} \mathrm{d}\sigma\, \sigma^{(1/d)-1} A_d \exp[-R\sigma(t-t^*)].$$

Also,

$$\Delta Q_d(t) \equiv Q_d(\infty) - Q_d(t) \underset{t\to\infty}{\simeq} (Rt)^{-1/d} A_d \int_0^\infty \mathrm{d}\hat{\sigma}\, \hat{\sigma}^{(1/d)-1}\, \mathrm{e}^{-\hat{\sigma}};$$

thus

$$\Delta Q_d(t) \underset{t\to\infty}{\simeq} A_d \Gamma(1/d)(Rt)^{-1/d} \tag{4}$$

where

$$A_d \equiv \frac{1}{d} \int \mathrm{d}\Theta\; p(2l|_-, \Theta) f(\Theta)^{-1/d}.$$

The asymptotic estimate (4) is in agreement with the one guessed from (1) and (2).

In one dimension, there are no angular variables, and one obtains

$$\Delta Q_1(t) \underset{t\to\infty}{\simeq} p(2l|_-)/(Rt) \tag{5}$$

where $p(\rho)\,\mathrm{d}\rho$ is the number density of holes of length between ρ and $\rho+\mathrm{d}\rho$. From equation (42) in Gonzáles *et al* (1974)

$$p(2l|_-) = \lim_{t\to\infty} t^2 \exp\left(-2 \int_0^t \frac{\mathrm{d}x}{x}(1-\mathrm{e}^{-x})\right).$$

As

$$\int_0^t \frac{\mathrm{d}x}{x}(1-\mathrm{e}^{-x}) \underset{t\to\infty}{\simeq} \ln t + \gamma + \ldots,$$

the asymptotic estimate obtained in (1) agrees with the one given in (5).

Another remark made by Feder (1979) is the divergence of the radial distribution function at contact in the final state, both in one and two dimensions. This can be explained by the previous sort of argument.

Let $G(r)$ be this radial distribution function. Given a car with its centre at 0, $G(r)\mathrm{d}r$ is the probability that another car has its centre between r and $r+\mathrm{d}r$ from this one. Consider now a 'fillable' small hole drawn around a VVDP. The car to be parked in this hole yields $(d+1)$ contributions to $G(r)$ in the range $r \sim 2l|_+$, as this hole is close to $(d+1)$ other cars. Actually, for a given hole this contributes to an angle-dependent pair contribution. This angular dependence vanishes either after averaging over all the holes or after a convenient angular integration, as we shall do tacitly.

Let $G(r|\rho, \Theta)$ be the contribution to $G(r)$ of holes in $\mathrm{d}\rho$, $\mathrm{d}\Theta$ around (ρ, Θ). A (ρ, Θ) hole contributes to $G(r)$ for values of r in the range $2l < r < 2l + \delta\phi(\Theta)$, where the factor $\phi(\Theta)$ accounts for the fact that the maximum distance from a point in the hole to one of the $(d+1)$ surrounding cars depends on Θ. Since all the holes are certainly filled, and contribute as a whole $(d+1)$ times to $G(r)$, one has

$$\int_{2l}^{2l+\delta\phi(\Theta)} \mathrm{d}r\, G(r|\rho, \Theta) = (d+1)p(\rho, \Theta), \tag{6}$$

$p(\rho, \Theta)$ being, as before, the probability distribution for the VVDP. This implies that near $r = 2l$ and $\rho = 0$, $G(r|\rho, \Theta)$ is of the form $\delta^{-1}\tilde{G}(\delta/\eta, \Theta)$, where we have put, for convenience, $r = 2l + \eta$. The function $\tilde{G}$ can be computed in principle once $p(\rho, \Theta)$ is given. Its main property here is that it is continuous at $\delta/\eta \to \infty$ so that the left-hand side of (6) has a finite limit as $\eta \to 0$.

Now the contribution of all the small holes to $G(r)$, say $G_{\mathrm{sh}}(r)$, is obtained by integration of $\delta^{-1}\tilde{G}(\delta/\eta, \Theta)$ over ρ and r:

$$G_{\mathrm{sh}}(r = 2l + \delta) = \int_{0<\delta<\delta^*} \mathrm{d}\Theta\, \mathrm{d}\delta\, \delta^{-1}\tilde{G}(\delta/\eta, \Theta).$$

The integrand vanishes for $\phi(\Theta)\delta < \eta$. Accordingly, for a given δ much smaller than δ^* this is a function of ρ that is of order $\delta^{-1}\tilde{G}(0, \Theta)$ as $\eta/\phi(\Theta) \ll \delta \ll \delta^*$, and has some structure in the region $\eta \sim \delta$. The contribution of the region $\eta/\phi(\Theta) \ll \delta \ll \delta^*$ is of order $\ln \eta$; this shows that the pair correlation function diverges logarithmically at contact, in the range $r \equiv 2l + \eta \to 2l|_+$. This logarithmic divergence can be made explicit in the 1D case, where (Feder 1979)

$$G(2l+\eta) \underset{\eta \to 0_+}{\simeq} -2l^{-2\lambda} \ln \eta,$$

λ again being Euler's constant. Again this direct estimate coincides with the one obtained by the above method.

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